LESSON 4-5 Proving Triangles Congruent—ASA, AAS

Then

- You proved triangles congruent using SSS and SAS.

Now

1. Use the ASA Postulate to test for congruence.
2. Use the AAS Theorem to test for congruence.

Why?

- Competitive sweep rowing, also called *crew*, involves two or more people who sit facing the stern of the boat, with each rower pulling one oar. In high school competitions, a race, called a *regatta*, usually requires a body of water that is more than 1500 meters long. Congruent triangles can be used to measure distances that are not easily measured directly, like the length of a regatta course.

Ben Blankenburg/CORBIS

New Vocabulary
included side

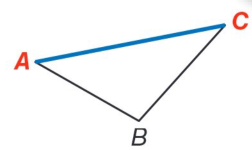
Common Core State Standards

Content Standards
G.CO.10 Prove theorems about triangles.
G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Mathematical Practices
3 Construct viable arguments and critique the reasoning of others.
5 Use appropriate tools strategically.

1 ASA Postulate

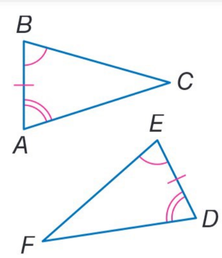
An **included side** is the side located between two consecutive angles of a polygon. In $\triangle ABC$ at the right, $\overline{AC}$ is the included side between $\angle A$ and $\angle C$.



Postulate 4.3 Angle-Side-Angle (ASA) Congruence

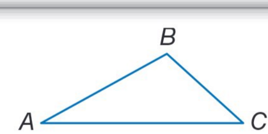
If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

Example If **A**ngle $\angle A \cong \angle D$,
Side $\overline{AB} \cong \overline{DE}$, and
Angle $\angle B \cong \angle E$,
then $\triangle ABC \cong \triangle DEF$.



Construction Congruent Triangles Using Two Angles and Included Side

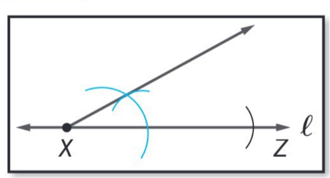
Draw a triangle and label it $\triangle ABC$. Then use the ASA Postulate to construct $\triangle XYZ \cong \triangle ABC$.



Step 1

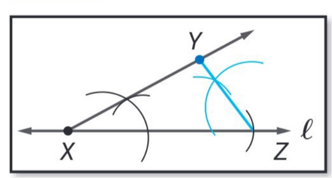
Draw a line ℓ and select a point X. Construct $\overline{XZ}$ such that $\overline{XZ} \cong \overline{AC}$.

Step 2



Construct an angle congruent to $\angle A$ at X using $\overleftrightarrow{XZ}$ as a side of the angle.

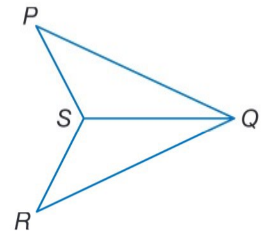
Step 3



Construct an angle congruent to $\angle C$ at Z using $\overleftrightarrow{XZ}$ as a side of the angle. Label the point where the new sides of the angles meet as Y.

Example 1 Use ASA to Prove Triangles Congruent

Write a two-column proof.



Given: $\overline{QS}$ bisects $\angle PQR$; $\angle PSQ \cong \angle RSQ$.

Prove: $\triangle PQS \cong \triangle RQS$

Proof:

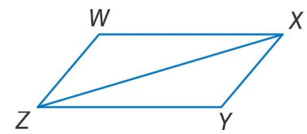
Statements	Reasons
1. $\overline{QS}$ bisects $\angle PQR$; $\angle PSQ \cong \angle RSQ$.	**1.** Given
2. $\angle PQS \cong \angle RQS$	**2.** Definition of Angle Bisector
3. $\overline{QS} \cong \overline{QS}$	**3.** Reflexive Property of Congruence
4. $\triangle PQS \cong \triangle RQS$	**4.** ASA

Guided Practice

1. Write a flow proof.

Given: $\overline{ZX}$ bisects $\angle WZY$; $\overline{XZ}$ bisects $\angle YXW$.

Prove: $\triangle WXZ \cong \triangle XZY$



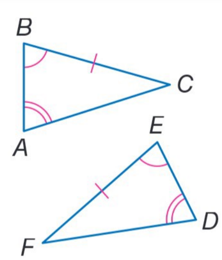
2 AAS Theorem

The congruence of two angles and a nonincluded side are also sufficient to prove two triangles congruent. This congruence relationship is a theorem because it can be proved using the Third Angles Theorem.

Theorem 4.5 Angle-Angle-Side (AAS) Congruence

If two angles and the nonincluded side of one triangle are congruent to the corresponding two angles and side of a second triangle, then the two triangles are congruent.

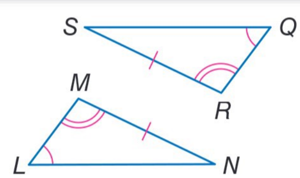
Example If **A**ngle $\angle A \cong \angle D$,
Angle $\angle B \cong \angle E$, and
Side $\overline{BC} \cong \overline{EF}$,
then $\triangle ABC \cong \triangle DEF$.



Proof Angle-Angle-Side Theorem

Given: $\angle L \cong \angle Q$, $\angle M \cong \angle R$, $\overline{MN} \cong \overline{RS}$

Prove: $\triangle LMN \cong \triangle QRS$



Proof:

$\boxed{\angle L \cong \angle Q}$ Given

$\boxed{\angle M \cong \angle R}$ Given

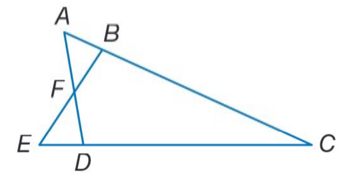
$\boxed{\overline{MN} \cong \overline{RS}}$ Given

$\boxed{\angle N \cong \angle S}$ Third Angles Theorem

$\boxed{\triangle LMN \cong \triangle QRS}$ AAS

Example 2 Use AAS to Prove Triangles Congruent

Write a two-column proof.

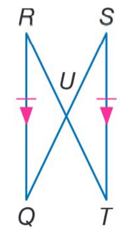


Given: $\angle DAC \cong \angle BEC$
$\overline{DC} \cong \overline{BC}$

Prove: $\triangle ACD \cong \triangle ECB$

Proof: We are given that $\angle DAC \cong \angle BEC$ and $\overline{DC} \cong \overline{BC}$. $\angle C \cong \angle C$ by the Reflexive Property. By AAS, $\triangle ACD \cong \triangle ECB$.

GuidedPractice

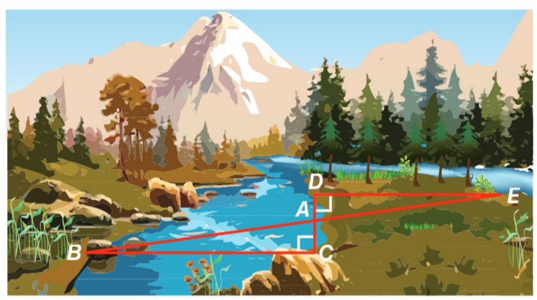


2. Write a flow proof.
Given: $\overline{RQ} \cong \overline{ST}$ and $\overline{RQ} \parallel \overline{ST}$
Prove: $\triangle RUQ \cong \triangle TUS$

You can use congruent triangles to measure distances that are difficult to measure directly.

Real-World Example 3 Apply Triangle Congruence

COMMUNITY SERVICE **Jeremias is working with a community service group to build a bridge across a creek at a local park. The bridge will span the creek between points *C* and *B*. Jeremias located a fixed point *D* to use as a reference point so that the segments have the relationships shown. *A* is the midpoint of $\overline{CD}$ and *DE* is 15 feet. How long does the bridge need to be?**



In order to determine the length of $\overline{CB}$, we must first prove that the two triangles Jeremias has created are congruent.

- Since $\overline{CD}$ is perpendicular to both $\overline{CB}$ and $\overline{DE}$, the segments form right angles as shown on the diagram.
- All right angles are congruent, so $\angle BCA \cong \angle EDA$.
- Point A is the midpoint of $\overline{CD}$, so $\overline{CA} \cong \overline{AD}$.
- $\angle BAC$ and $\angle EAD$ are vertical angles, so they are congruent.

Therefore, by ASA, $\triangle BAC \cong \triangle EAD$.

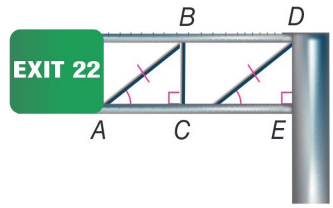
Since $\triangle BAC \cong \triangle EAD$, $\overline{DE} \cong \overline{CB}$ by CPCTC. Since the measure of $\overline{DE}$ is 15 feet, the measure of $\overline{CB}$ is also 15 feet. Therefore, the bridge needs to be 15 feet long.

StudyTip

Angle-Angle-Angle In Example 3, $\angle B$ and $\angle E$ are congruent by the Third Angles Theorem. Congruence of all three corresponding angles is not sufficient, however, to prove two triangles congruent.

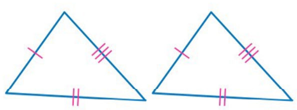
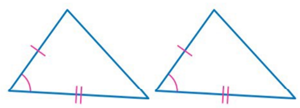
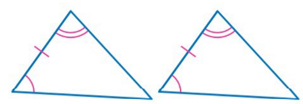
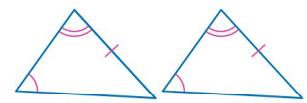
Guided Practice

3. In the sign scaffold shown at the right, $\overline{BC} \perp \overline{AC}$ and $\overline{DE} \perp \overline{CE}$. $\angle BAC \cong \angle DCE$, and $\overline{AB} \cong \overline{CD}$. Write a paragraph proof to show that $\overline{BC} \cong \overline{DE}$.



You have learned several methods for proving triangle congruence.

Concept Summary Proving Triangles Congruent

SSS	SAS	ASA	AAS
			
Three pairs of corresponding sides are congruent.	Two pairs of corresponding sides and their included angles are congruent.	Two pairs of corresponding angles and their included sides are congruent.	Two pairs of corresponding angles and the corresponding nonincluded sides are congruent.

Check Your Understanding

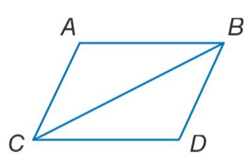
= Step-by-Step Solutions begin on page R14.

Example 1 **PROOF** **Write the specified type of proof.**

1. two-column proof

Given: $\overline{CB}$ bisects $\angle ABD$ and $\angle ACD$.

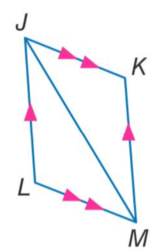
Prove: $\triangle ABC \cong \triangle DBC$



2. flow proof

Given: $\overline{JK} \parallel \overline{LM}$, $\overline{JL} \parallel \overline{KM}$

Prove: $\triangle JML \cong \triangle MJK$

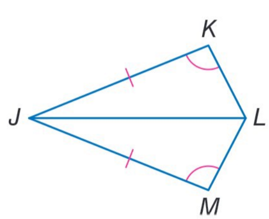


Example 2

3. paragraph proof

Given: $\angle K \cong \angle M$, $\overline{JK} \cong \overline{JM}$, $\overline{JL}$ bisects $\angle KLM$.

Prove: $\triangle JKL \cong \triangle JML$

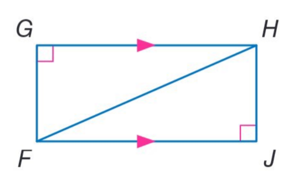


4. two-column proof

Given: $\overline{GH} \parallel \overline{FJ}$

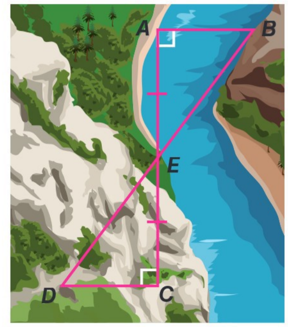
$m\angle G = m\angle J = 90$

Prove: $\triangle HJF \cong \triangle FGH$



Example 3

5 **BRIDGE BUILDING** A surveyor needs to find the distance from point A to point B across a canyon. She places a stake at A, and a coworker places a stake at B on the other side of the canyon. The surveyor then locates C on the same side of the canyon as A such that $\overline{CA} \perp \overline{AB}$. A fourth stake is placed at E, the midpoint of $\overline{CA}$. Finally, a stake is placed at D such that $\overline{CD} \perp \overline{CA}$ and D, E, and B are sited as lying along the same line.



a. Explain how the surveyor can use the triangles formed to find AB.

b. If $AC = 1300$ meters, $DC = 550$ meters, and $DE = 851.5$ meters, what is AB? Explain your reasoning.

Practice and Problem Solving

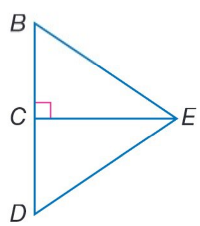
Extra Practice is on page R4.

Example 1

PROOF **Write a paragraph proof.**

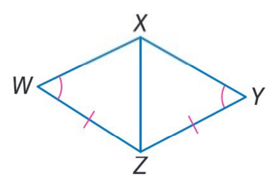
6. **Given:** $\overline{CE}$ bisects $\angle BED$; $\angle BCE$ and $\angle ECD$ are right angles.

Prove: $\triangle ECB \cong \triangle ECD$

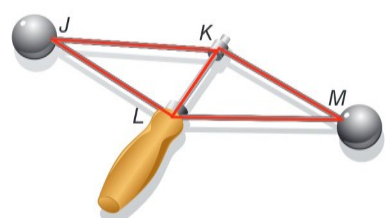


7. **Given:** $\angle W \cong \angle Y$, $\overline{WZ} \cong \overline{YZ}$, $\overline{XZ}$ bisects $\angle WZY$.

Prove: $\triangle XWZ \cong \triangle XYZ$



8. **TOYS** The object of the toy shown is to make the two spheres meet and strike each other repeatedly on one side of the wand and then again on the other side. If $\angle JKL \cong \angle MLK$ and $\angle JLK \cong \angle MKL$, prove that $\overline{JK} \cong \overline{ML}$.

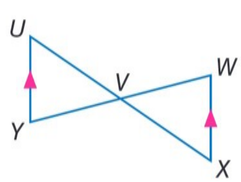


Example 2

PROOF **Write a two-column proof.**

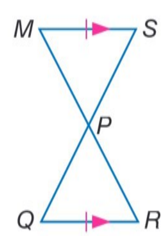
9 **Given:** V is the midpoint of $\overline{YW}$; $\overline{UY} \parallel \overline{XW}$.

Prove: $\triangle UVY \cong \triangle XVW$



10. **Given:** $\overline{MS} \cong \overline{RQ}$, $\overline{MS} \parallel \overline{RQ}$

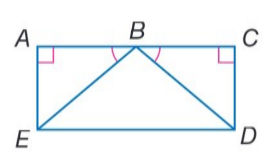
Prove: $\triangle MSP \cong \triangle RQP$



11. **CCSS** **ARGUMENTS** Write a flow proof.

Given: $\angle A$ and $\angle C$ are right angles. $\angle ABE \cong \angle CBD$, $\overline{AE} \cong \overline{CD}$

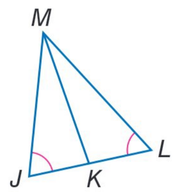
Prove: $\overline{BE} \cong \overline{BD}$



12. PROOF Write a flow proof.

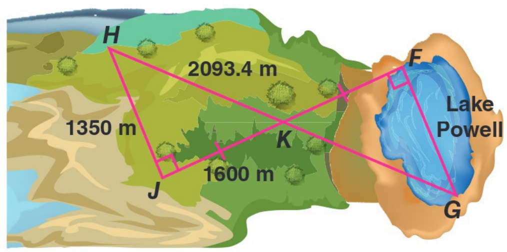
Given: $\overline{KM}$ bisects $\angle JML$; $\angle J \cong \angle L$.

Prove: $\overline{JM} \cong \overline{LM}$



Example 3

13. CCSS MODELING A high school wants to hold a 1500-meter regatta on Lake Powell but is unsure if the lake is long enough. To measure the distance across the lake, the crew members locate the vertices of the triangles below and find the measures of the lengths of $\triangle HJK$ as shown below.

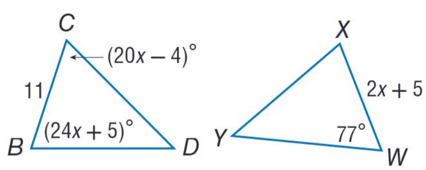


a. Explain how the crew team can use the triangles formed to estimate the distance FG across the lake.

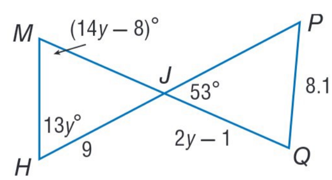
b. Using the measures given, is the lake long enough for the team to use as the location for their regatta? Explain your reasoning.

ALGEBRA Find the value of the variable that yields congruent triangles.

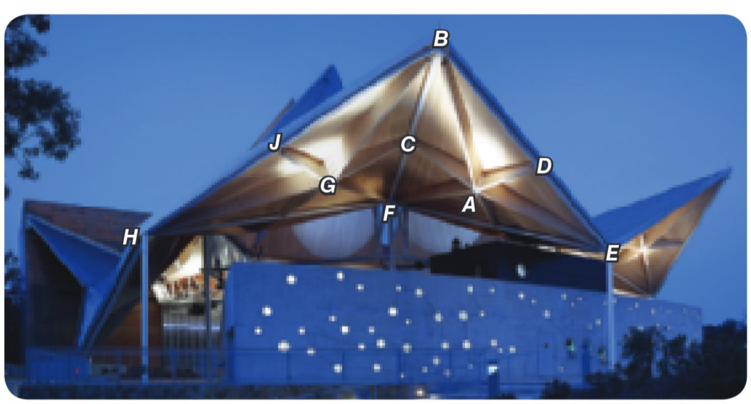
14. $\triangle BCD \cong \triangle WXY$



15 $\triangle MHJ \cong \triangle PQJ$



16. THEATER DESIGN The trusses of the roof of the outdoor theater shown below appear to be several different pairs of congruent triangles. Assume that trusses that appear to lie on the same line actually lie on the same line.



Greg Murphey

a. If $\overline{AB}$ bisects $\angle CBD$ and $\angle CAD$, prove that $\triangle ABC \cong \triangle ABD$.

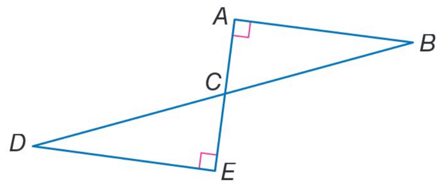
b. If $\triangle ABC \cong \triangle ABD$ and $\angle FCA \cong \angle EDA$, prove that $\triangle CAF \cong \triangle DAE$.

c. If $\overline{HB} \cong \overline{EB}$, $\angle BHG \cong \angle BEA$, $\angle HGJ \cong \angle EAD$, and $\angle JGB \cong \angle DAB$, prove that $\triangle BHG \cong \triangle BEA$.

PROOF **Write a paragraph proof.**

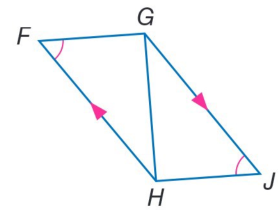
17. Given: $\overline{AE} \perp \overline{DE}$, $\overline{EA} \perp \overline{AB}$, C is the midpoint of $\overline{AE}$.

Prove: $\overline{CD} \cong \overline{CB}$



18. Given: $\angle F \cong \angle J$, $\overline{FH} \parallel \overline{GJ}$

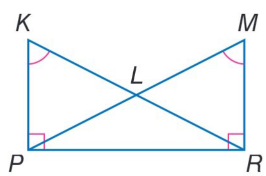
Prove: $\overline{FH} \cong \overline{JG}$



PROOF **Write a two-column proof.**

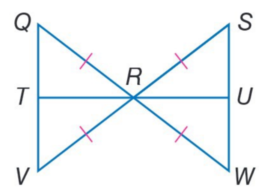
19. Given: $\angle K \cong \angle M$, $\overline{KP} \perp \overline{PR}$, $\overline{MR} \perp \overline{PR}$

Prove: $\angle KPL \cong \angle MRL$

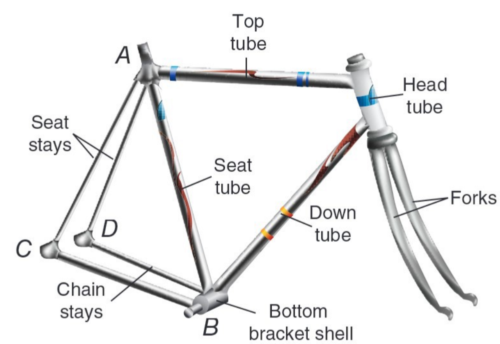


20. Given: $\overline{QR} \cong \overline{SR} \cong \overline{WR} \cong \overline{VR}$

Prove: $\overline{QT} \cong \overline{WU}$



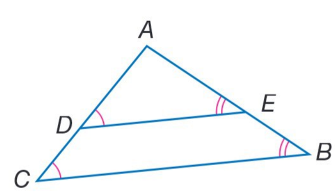
21 **FITNESS** The seat tube of a bicycle forms a triangle with each seat and chain stay as shown. If each seat stay makes a 44° angle with its corresponding chain stay and each chain stay makes a 68° angle with the seat tube, show that the two seat stays are the same length.



H.O.T. Problems Use Higher-Order Thinking Skills

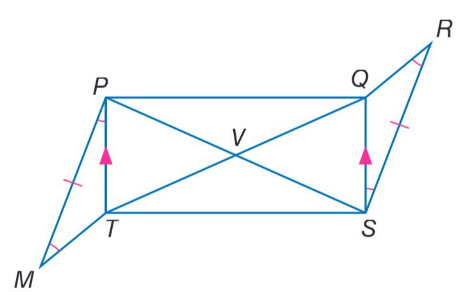
22. OPEN ENDED Draw and label two triangles that could be proved congruent by ASA.

23. CCSS CRITIQUE Tyrone says it is not possible to show that $\triangle ADE \cong \triangle ACB$. Lorenzo disagrees, explaining that since $\angle ADE \cong \angle ACB$, and $\angle A \cong \angle A$ by the Reflexive Property, $\triangle ADE \cong \triangle ACB$. Is either of them correct? Explain.



24. REASONING Find a counterexample to show why SSA (Side-Side-Angle) cannot be used to prove the congruence of two triangles.

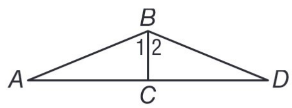
25. CHALLENGE Using the information given in the diagram, write a flow proof to show that $\triangle PVQ \cong \triangle SVT$.



26. WRITING IN MATH How do you know what method (SSS, SAS, etc.) to use when proving triangle congruence? Use a chart to explain your reasoning.

Standardized Test Practice

27. Given: $\overline{BC}$ is perpendicular to $\overline{AD}$; $\angle 1 \cong \angle 2$.



Which theorem or postulate could be used to prove $\triangle ABC \cong \triangle DBC$?

A AAS
B ASA
C SAS
D SSS

28. SHORT RESPONSE Write an expression that can be used to find the values of $s(n)$ in the table.

n	−8	−4	−1	0	1
$s(n)$	1.00	2.00	2.75	3.00	3.25

29. ALGEBRA If −7 is multiplied by a number greater than 1, which of the following describes the result?

F a number greater than 7
G a number between −7 and 7
H a number greater than −7
J a number less than −7

30. SAT/ACT $\sqrt{121 + 104} = ?$

A 15
B 21
C 25
D 125
E 225

Spiral Review

Determine whether $\triangle ABC \cong \triangle XYZ$. Explain. (Lesson 4-4)

31. $A(6, 4)$, $B(1, -6)$, $C(-9, 5)$, $X(0, 7)$, $Y(5, -3)$, $Z(15, 8)$

32. $A(0, 5)$, $B(0, 0)$, $C(-2, 0)$, $X(4, 8)$, $Y(4, 3)$, $Z(6, 3)$

33. ALGEBRA If $\triangle RST \cong \triangle JKL$, $RS = 7$, $ST = 5$, $RT = 9 + x$, $JL = 2x - 10$, and $JK = 4y - 5$, draw and label a figure to represent the congruent triangles. Then find x and y. (Lesson 4-3)

34. FINANCIAL LITERACY Maxine charges \$5 to paint a mailbox and \$4 per hour to mow a lawn. Write an equation to represent the amount of money Maxine can earn from a homeowner who has his or her mailbox painted and lawn mowed. (Lesson 3-4)

Copy and complete each truth table. (Lesson 2-2)

35.

p	q	$\sim p$	$\sim p \vee q$
F	T		
T	T		
F	F		
T	F		

36.

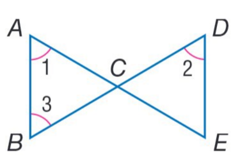
p	q	$\sim q$	$\sim q \wedge p$
F		F	
T		T	
T		F	
F		T	

Skills Review

PROOF **Write a two-column proof for each of the following.**

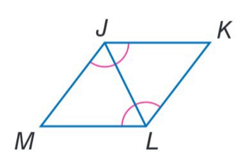
37. Given: $\angle 2 \cong \angle 1$
$\angle 1 \cong \angle 3$

Prove: $\overline{AB} \parallel \overline{DE}$



38. Given: $\angle MJK \cong \angle KLM$
$\angle LMJ$ and $\angle KLM$ are supplementary.

Prove: $\overline{KJ} \parallel \overline{LM}$



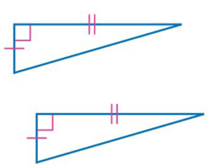
Geometry Lab
Congruence in Right Triangles

In Lessons 4-4 and 4-5, you learned theorems and postulates to prove triangles congruent. How do these theorems and postulates apply to right triangles?

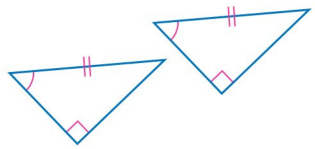
CCSS Common Core State Standards
Content Standards
G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
Mathematical Practices 5

Study each pair of right triangles.

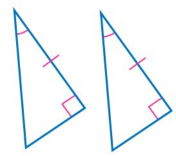
a.



b.



c.



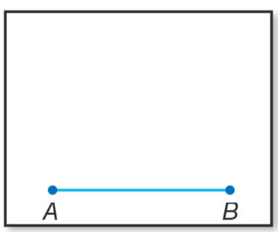
Analyze

1. Is each pair of triangles congruent? If so, which congruence theorem or postulate applies?
2. Rewrite the congruence rules from Exercise 1 using *leg*, (L), or *hypotenuse,* (H), to replace *side*. Omit the *A* for any right angle since we know that all right triangles contain a right angle and all right angles are congruent.
3. **MAKE A CONJECTURE** If you know that the corresponding legs of two right triangles are congruent, what other information do you need to declare the triangles congruent? Explain.

In Lesson 4-5, you learned that SSA is not a valid test for determining triangle congruence. Can SSA be used to prove right triangles congruent?

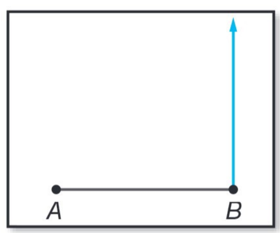
Activity SSA and Right Triangles

Step 1



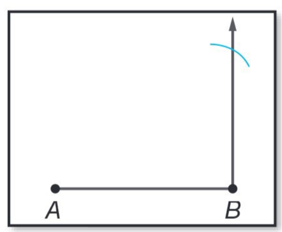
Draw $\overline{AB}$ so that $AB = 6$ centimeters.

Step 2



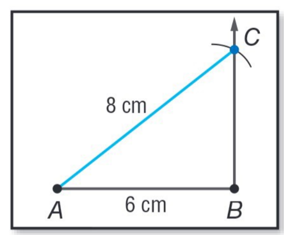
Use a protractor to draw a ray from B that is perpendicular to $\overline{AB}$.

Step 3



Open your compass to a width of 8 centimeters. Place the point at A and draw an arc to intersect the ray.

Step 4



Label the intersection C and draw $\overline{AC}$ to complete $\triangle ABC$.

Analyze

4. Does the model yield a unique triangle?
5. Can you use the lengths of the hypotenuse and a leg to show right triangles are congruent?
6. **Make a conjecture** about the case of SSA that exists for right triangles.

(continued on the next page)

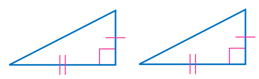
Geometry Lab
Congruence in Right Triangles *Continued*

Your work on the previous page provides evidence for four ways to prove right triangles congruent.

Theorem Right Triangle Congruence

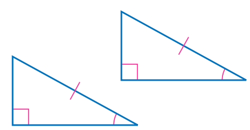
Theorem 4.6 Leg-Leg Congruence
If the legs of one right triangle are congruent to the corresponding legs of another right triangle, then the triangles are congruent.

Abbreviation *LL*



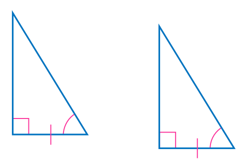
Theorem 4.7 Hypotenuse-Angle Congruence
If the hypotenuse and acute angle of one right triangle are congruent to the hypotenuse and corresponding acute angle of another right triangle, then the two triangles are congruent.

Abbreviation *HA*



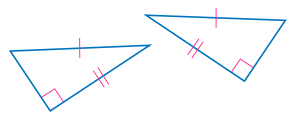
Theorem 4.8 Leg-Angle Congruence
If one leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another right triangle, then the triangles are congruent.

Abbreviation *LA*



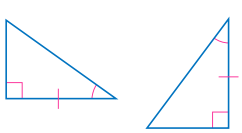
Theorem 4.9 Hypotenuse-Leg Congruence
If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the triangles are congruent.

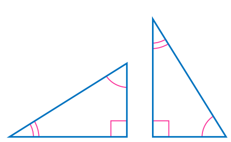
Abbreviation *HL*

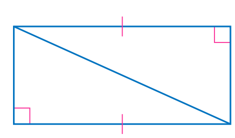


Exercises

Determine whether each pair of triangles is congruent. If yes, tell which postulate or theorem applies.

7. 

8. 

9. 

PROOF **Write a proof for each of the following.**

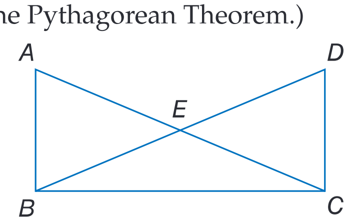
10. Theorem 4.6

11. Theorem 4.7

12. Theorem 4.8 (*Hint*: There are two possible cases.)

13. Theorem 4.9 (*Hint*: Use the Pythagorean Theorem.)

Use the figure at the right.



14. Given: $\overline{AB} \perp \overline{BC}$, $\overline{DC} \perp \overline{BC}$
$\overline{AC} \cong \overline{BD}$

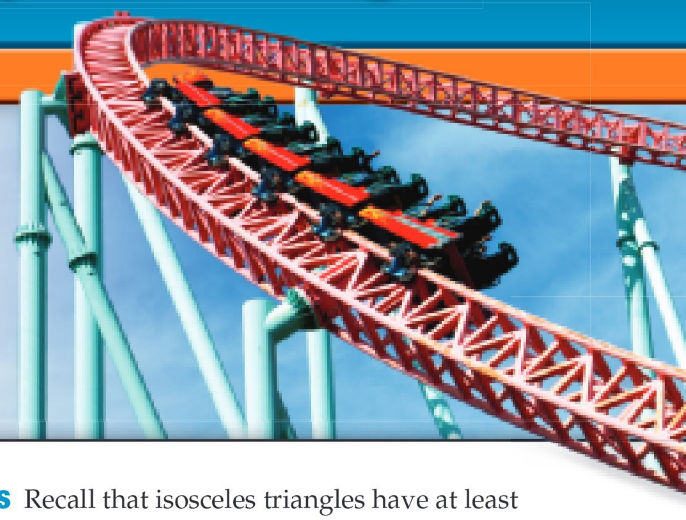
Prove: $\overline{AB} \cong \overline{DC}$

15. Given: $\overline{AB} \parallel \overline{DC}$, $\overline{AB} \perp \overline{BC}$
E is the midpoint of $\overline{AC}$ and $\overline{BD}$.

Prove: $\overline{AC} \cong \overline{DB}$

LESSON 4-6

Isosceles and Equilateral Triangles



:: Then	:: Now	:: Why?
• You identified isosceles and equilateral triangles.	• 1 Use properties of isosceles triangles. 2 Use properties of equilateral triangles.	• The tracks on the roller coaster have triangular reinforcements between the tracks for support and stability. The triangle supports in the photo are isosceles triangles.

NewVocabulary
legs of an isosceles triangle
vertex angle
base angles

Common Core State Standards

Content Standards

G.CO.10 Prove theorems about triangles.

G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

Mathematical Practices

2 Reason abstractly and quantitatively.

3 Construct viable arguments and critique the reasoning of others.

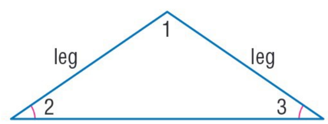
1 Properties of Isosceles Triangles

Recall that isosceles triangles have at least two congruent sides. The parts of an isosceles triangle have special names.

The two congruent sides are called the **legs of an isosceles triangle**, and the angle with sides that are the legs is called the **vertex angle**. The side of the triangle opposite the vertex angle is called the *base*. The two angles formed by the base and the congruent sides are called the **base angles**.

$\angle 1$ is the vertex angle.

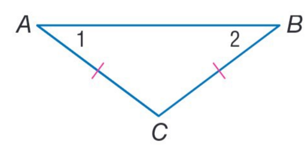
$\angle 2$ and $\angle 3$ are the base angles.



Theorems Isosceles Triangle

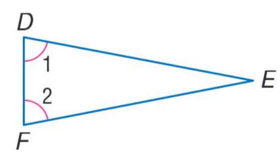
4.10 Isosceles Triangle Theorem
If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

Example If $\overline{AC} \cong \overline{BC}$, then $\angle 2 \cong \angle 1$.



4.11 Converse of Isosceles Triangle Theorem
If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

Example If $\angle 1 \cong \angle 2$, then $\overline{FE} \cong \overline{DE}$.



You will prove Theorem 4.11 in Exercise 37.

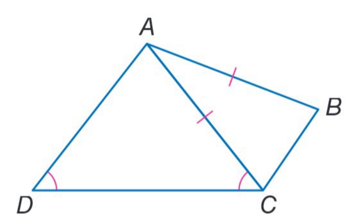
Example 1 Congruent Segments and Angles

a. Name two unmarked congruent angles.

$\angle ACB$ is opposite $\overline{AB}$ and $\angle B$ is opposite $\overline{AC}$, so $\angle ACB \cong \angle B$.

b. Name two unmarked congruent segments.

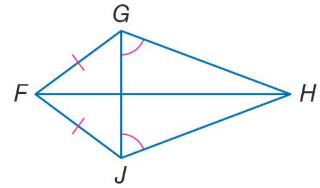
$\overline{AD}$ is opposite $\angle ACD$ and $\overline{AC}$ is opposite $\angle D$, so $\overline{AD} \cong \overline{AC}$.



GuidedPractice

1A. Name two unmarked congruent angles.

1B. Name two unmarked congruent segments.

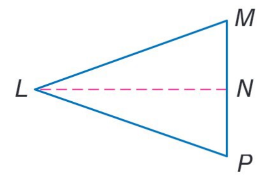


To prove the Isosceles Triangle Theorem, draw an auxiliary line and use the two triangles formed.

Proof Isosceles Triangle Theorem

Given: $\triangle LMP$; $\overline{LM} \cong \overline{LP}$

Prove: $\angle M \cong \angle P$



Proof:

Statements	Reasons
1. Let N be the midpoint of $\overline{MP}$.	**1.** Every segment has exactly one midpoint.
2. Draw an auxiliary segment $\overline{LN}$.	**2.** Two points determine a line.
3. $\overline{MN} \cong \overline{PN}$	**3.** Midpoint Theorem
4. $\overline{LN} \cong \overline{LN}$	**4.** Reflexive Property of Congruence
5. $\overline{LM} \cong \overline{LP}$	**5.** Given
6. $\triangle LMN \cong \triangle LPN$	**6.** SSS
7. $\angle M \cong \angle P$	**7.** CPCTC

2 Properties of Equilateral Triangles

The Isosceles Triangle Theorem leads to two corollaries about the angles of an equilateral triangle.

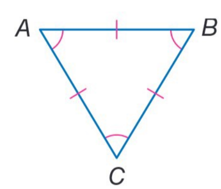
ReviewVocabulary

equilateral triangle a triangle with three congruent sides

Corollaries Equilateral Triangle

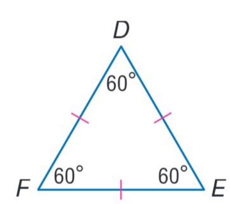
4.3 A triangle is equilateral if and only if it is equiangular.

Example If $\angle A \cong \angle B \cong \angle C$, then $\overline{AB} \cong \overline{BC} \cong \overline{CA}$.



4.4 Each angle of an equilateral triangle measures 60.

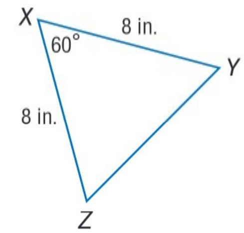
Example If $\overline{DE} \cong \overline{EF} \cong \overline{FE}$, then $m\angle A = m\angle B = m\angle C = 60$.



You will prove Corollaries 4.3 and 4.4 in Exercises 35 and 36.

Example 2 Find Missing Measures

Find each measure.



a. $m\angle Y$

Since $XY = XZ$, $\overline{XY} \cong \overline{XZ}$. By the Isosceles Triangle Theorem, base angles Z and Y are congruent, so $m\angle Z = m\angle Y$. Use the Triangle Sum Theorem to write and solve an equation to find $m\angle Y$.

$m\angle X + m\angle Y + m\angle Z = 180$	Triangle Sum Theorem
$60 + m\angle Y + m\angle Y = 180$	$m\angle X = 60$, $m\angle Z = m\angle Y$
$60 + 2(m\angle Y) = 180$	Simplify.
$2(m\angle Y) = 120$	Subtract 60 from each side.
$m\angle Y = 60$	Divide each side by 2.

b. YZ

$m\angle Z = m\angle Y$, so $m\angle Z = 60$ by substitution. Since $m\angle X = 60$, all three angles measure 60, so the triangle is equiangular. Because an equiangular triangle is also equilateral, $XY = XZ = ZY$. Since $XY = 8$ inches, $YZ = 8$ inches by substitution.

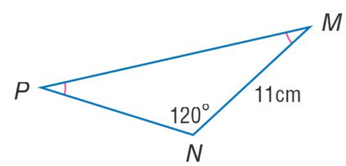
Study Tip

Isosceles Triangles As you discovered in Example 2, any isosceles triangle that has one 60° angle must be an equilateral triangle.

Guided Practice

2A. $m\angle M$

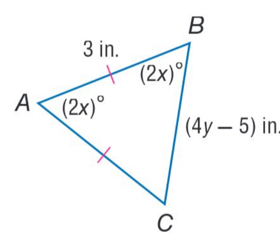
2B. PN



You can use the properties of equilateral triangles and algebra to find missing values.

Example 3 Find Missing Values

ALGEBRA Find the value of each variable.



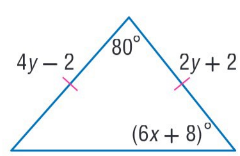
Since $\angle B = \angle A$, $\overline{AC} \cong \overline{BC}$ by the Converse of the Isosceles Triangle Theorem. All of the sides of the triangle are congruent, so the triangle is equilateral. Each angle of an equilateral triangle measures 60°, so $2x = 60$ and $x = 30$.

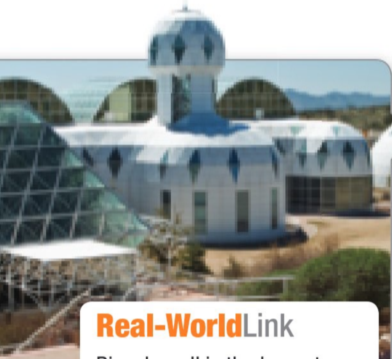
The triangle is equilateral, so all of the sides are congruent, and the lengths of all of the sides are equal.

$AB = BC$	Definition of equilateral triangle
$3 = 4y - 5$	Substitution
$8 = 4y$	Add 5 to each side.
$2 = y$	Divide each side by 4.

Guided Practice

3. Find the value of each variable.





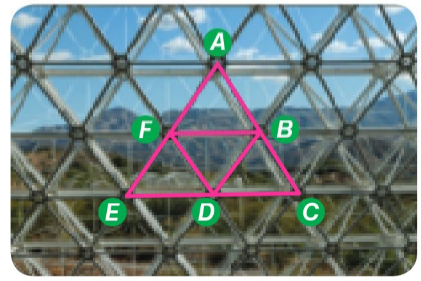
Real-WorldLink

Biosphere II is the largest totally enclosed ecosystem ever built, covering 3.14 acres in Oracle, Arizona. The controlled-environment facility is 91 feet at its highest point, and it has 6500 windows that enclose a volume of 7.2 million cubic feet.

Source: University of Arizona

Real-World Example 4 Apply Triangle Congruence

ENVIRONMENT **Refer to the photo of Biosphere II at the right. $\triangle ACE$ is an equilateral triangle. F is the midpoint of $\overline{AE}$, D is the midpoint of $\overline{EC}$, and B is the midpoint of $\overline{CA}$. Prove that $\triangle FBD$ is also equilateral.**



Given: $\triangle ACE$ is equilateral. F is the midpoint of $\overline{AE}$, D is the midpoint of $\overline{EC}$, and B is the midpoint of $\overline{CA}$.

Prove: $\triangle FBD$ is equilateral.

Proof:

Statements	Reasons
1. $\triangle ACE$ is equilateral.	**1.** Given
2. F is the midpoint of AE, D is the midpoint of EC, and B is the midpoint of CA.	**2.** Given
3. $m\angle A = 60$, $m\angle C = 60$, $m\angle E = 60$	**3.** Each angle of an equilateral triangle measures 60.
4. $\angle A \cong \angle C \cong \angle E$	**4.** Definition of congruence and substitution
5. $\overline{AE} \cong \overline{EC} \cong \overline{CA}$	**5.** Definition of equilateral triangle
6. $AE = EC = CA$	**6.** Definition of congruence
7. $\overline{AF} \cong \overline{FE}$, $\overline{ED} \cong \overline{DC}$, $\overline{CB} \cong \overline{BA}$	**7.** Midpoint Theorem
8. $AF = FE$, $ED = DC$, $CB = BA$	**8.** Definition of congruence
9. $AF + FE = AE$, $ED + DC = EC$, $CB + BA = CA$	**9.** Segment Addition Postulate
10. $AF + AF = AE$, $FE + FE = AE$, $ED + ED = EC$, $DC + DC = EC$, $CB + CB = CA$, $BA + BA = CA$	**10.** Substitution
11. $2AF = AE$, $2FE = AE$, $2ED = EC$, $2DC = EC$, $2CB = CA$, $2BA = CA$	**11.** Addition Property
12. $2AF = AE$, $2FE = AE$, $2ED = AE$, $2DC = AE$, $2CB = AE$, $2BA = AE$	**12.** Substitution Property
13. $2AF = 2ED = 2CB$, $2FE = 2DC = 2BA$	**13.** Transitive Property
14. $AF = ED = CB$, $FE = DC = BA$	**14.** Division Property
15. $\overline{AF} \cong \overline{ED} \cong \overline{CB}$, $\overline{FE} \cong \overline{DC} \cong \overline{BA}$	**15.** Definition of congruence
16. $\triangle AFB \cong \triangle EDF \cong \triangle CBD$	**16.** SAS
17. $\overline{DF} \cong \overline{FB} \cong \overline{BD}$	**17.** CPCTC
18. $\triangle FBD$ is equilateral.	**18.** Definition of equilateral triangle

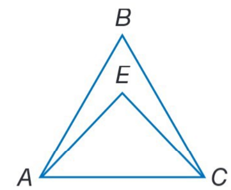
GuidedPractice

4. Given that $\triangle ACE$ is equilateral, $\overline{FB} \parallel \overline{EC}$, $\overline{FD} \parallel \overline{BC}$, $\overline{BD} \parallel \overline{EF}$, and D is the midpoint of $\overline{EC}$, prove that $\triangle FED \cong \triangle BDC$.

Check Your Understanding

= Step-by-Step Solutions begin on page R14.

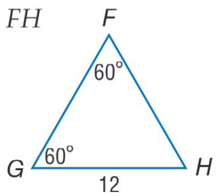
Example 1 **Refer to the figure at the right.**



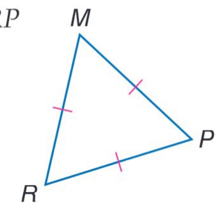
1. If $\overline{AB} \cong \overline{CB}$, name two congruent angles.
2. If $\angle EAC \cong \angle ECA$, name two congruent segments.

Example 2 **Find each measure.**

3. FH

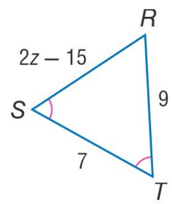


4. $m\angle MRP$

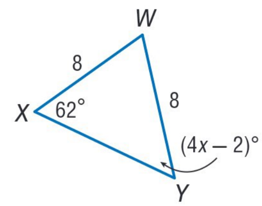


Example 3 CCSS **SENSE-MAKING** **Find the value of each variable.**

5.



6.

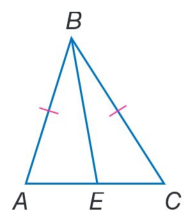


Example 4

7. **PROOF** **Write a two-column proof.**

   Given: $\triangle ABC$ is isosceles; $\overline{EB}$ bisects $\angle ABC$.

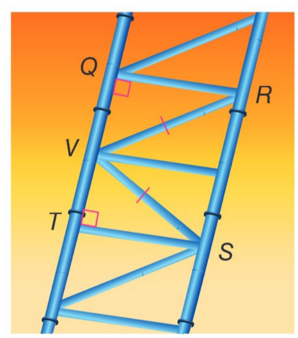
   Prove: $\triangle ABE \cong \triangle CBE$



8. **ROLLER COASTERS** The roller coaster track shown in the photo on page 285 appears to be composed of congruent triangles. A portion of the track is shown.

   a. If $\overline{QR}$ and $\overline{ST}$ are perpendicular to $\overline{QT}$, $\triangle VSR$ is isosceles with base $\overline{SR}$, and $\overline{QT} \parallel \overline{SR}$, prove that $\triangle RQV \cong \triangle STV$.

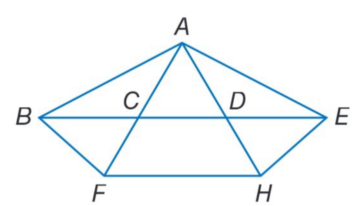
   b. If $VR = 2.5$ meters and $QR = 2$ meters, find the distance between $\overline{QR}$ and $\overline{ST}$. Explain your reasoning.



Practice and Problem Solving

Extra Practice is on page R4.

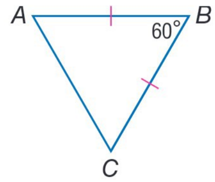
Example 1 **Refer to the figure at the right.**



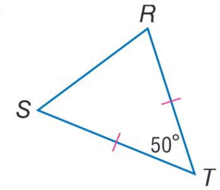
9. If $\overline{AB} \cong \overline{AE}$, name two congruent angles.
10. If $\angle ABF \cong \angle AFB$, name two congruent segments.
11. If $\overline{CA} \cong \overline{DA}$, name two congruent angles.
12. If $\angle DAE \cong \angle DEA$, name two congruent segments.
13. If $\angle BCF \cong \angle BFC$, name two congruent segments.
14. If $\overline{FA} \cong \overline{AH}$, name two congruent angles.

Example 2 **Find each measure.**

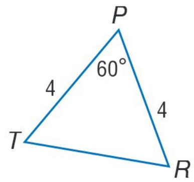
15. $m\angle BAC$



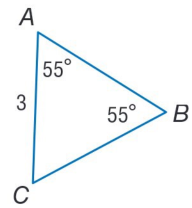
16. $m\angle SRT$



17. TR

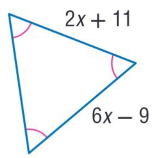


18. CB

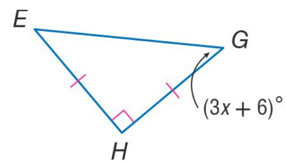


Example 3 **CCSS REGULARITY** **Find the value of each variable.**

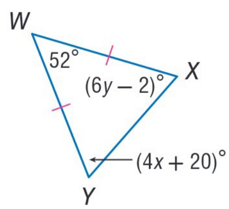
19.



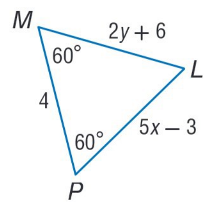
20.



21.



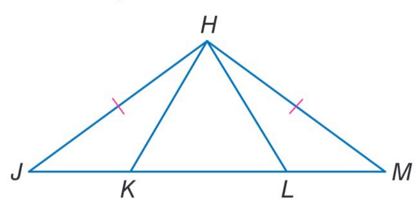
22.



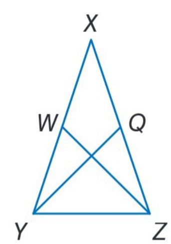
Example 4 **PROOF** **Write a paragraph proof.**

23. **Given:** $\triangle HJM$ is isosceles, and $\triangle HKL$ is equilateral. $\angle JKH$ and $\angle HKL$ are supplementary and $\angle HLK$ and $\angle MLH$ are supplementary.

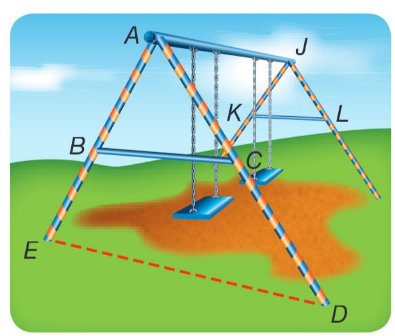
Prove: $\angle JHK \cong \angle MHL$



24. **Given:** $\overline{XY} \cong \overline{XZ}$
W is the midpoint of $\overline{XY}$.
Q is the midpoint of $\overline{XZ}$.
Prove: $\overline{WZ} \cong \overline{QY}$



25. **BABYSITTING** While babysitting her neighbor's children, Elisa observes that the supports on either side of a park swing set form two sets of triangles. Using a jump rope to measure, Elisa is able to determine that $\overline{AB} \cong \overline{AC}$, but $\overline{BC} \ncong \overline{AB}$.



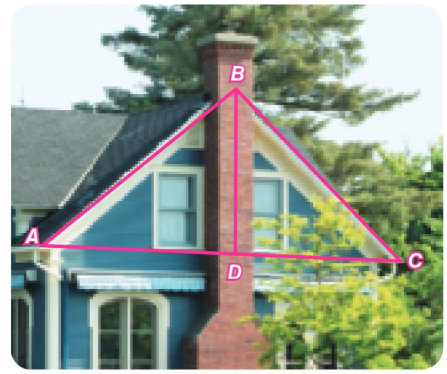
a. Elisa estimates $m\angle BAC$ to be 50. Based on this estimate, what is $m\angle ABC$? Explain.

b. If $\overline{BE} \cong \overline{CD}$, show that $\triangle AED$ is isosceles.

c. If $\overline{BC} \parallel \overline{ED}$ and $\overline{ED} \cong \overline{AD}$, show that $\triangle AED$ is equilateral.

d. If $\triangle JKL$ is isosceles, what is the minimum information needed to prove that $\triangle ABC \cong \triangle JLK$? Explain your reasoning.

26. **CHIMNEYS** In the picture, $\overline{BD} \perp \overline{AC}$ and $\triangle ABC$ is an isosceles triangle with base $\overline{AC}$. Show that the chimney of the house, represented by $\overline{BD}$, bisects the angle formed by the sloped sides of the roof, $\angle ABC$.



27. **CONSTRUCTION** Construct three different isosceles right triangles. Explain your method. Then verify your constructions using measurement and mathematics.

28. **PROOF** Based on your construction in Exercise 27, make and prove a conjecture about the relationship between the base angles of an isosceles right triangle.

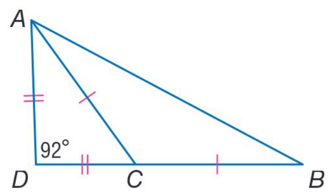
CCSS REGULARITY **Find each measure.**

29. $m\angle CAD$

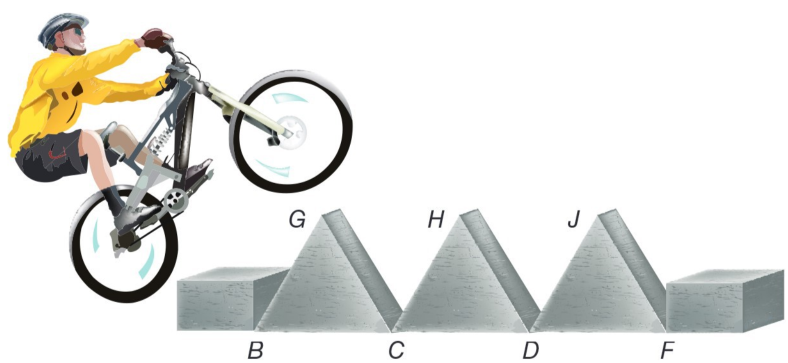
30. $m\angle ACD$

31. $m\angle ACB$

32. $m\angle ABC$

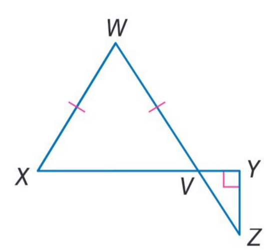


33. **FITNESS** In the diagram, the rider will use his bike to hop across the tops of each of the concrete solids shown. If each triangle is isosceles with vertex angles G, H, and J, and $\overline{BG} \cong \overline{HC}$, $\overline{HD} \cong \overline{JF}$, $\angle G \cong \angle H$, and $\angle H \cong \angle J$, show that the distance from B to F is three times the distance from D to F.



34. **Given:** $\triangle XWV$ is isosceles; $\overline{ZY} \perp \overline{YV}$.

Prove: $\angle X$ and $\angle YZV$ are complementary.

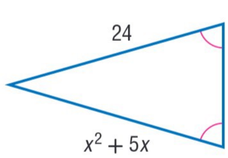


PROOF **Write a two-column proof of each corollary or theorem.**

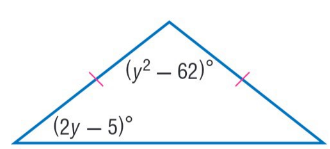
35. Corollary 4.3 **36.** Corollary 4.4 **37.** Theorem 4.11

Find the value of each variable.

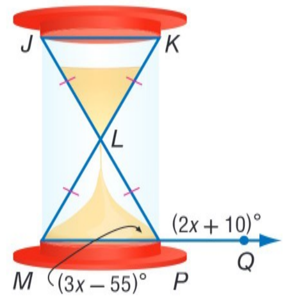
38.



39.



GAMES Use the diagram of a game timer shown to find each measure.



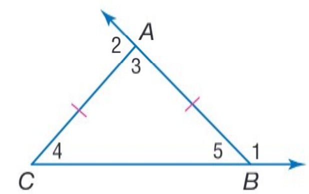
40. $m\angle LPM$

41. $m\angle LMP$

42. $m\angle JLK$

43. $m\angle JKL$

44. **MULTIPLE REPRESENTATIONS** In this problem, you will explore possible measures of the interior angles of an isosceles triangle given the measure of one exterior angle.



a. **Geometric** Use a ruler and a protractor to draw three different isosceles triangles, extending one of the sides adjacent to the vertex angle and to one of the base angles, and labeling as shown.

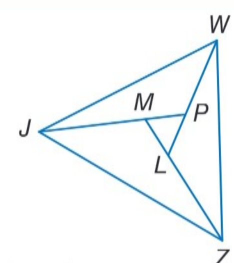
b. **Tabular** Use a protractor to measure and record $m\angle 1$ for each triangle. Use $m\angle 1$ to calculate the measures of $\angle 3$, $\angle 4$, and $\angle 5$. Then find and record $m\angle 2$ and use it to calculate these same measures. Organize your results in two tables.

c. **Verbal** Explain how you used $m\angle 1$ to find the measures of $\angle 3$, $\angle 4$, and $\angle 5$. Then explain how you used $m\angle 2$ to find these same measures.

d. **Algebraic** If $m\angle 1 = x$, write an expression for the measures of $\angle 3$, $\angle 4$, and $\angle 5$. Likewise, if $m\angle 2 = x$, write an expression for these same angle measures.

H.O.T. Problems Use Higher-Order Thinking Skills

45. **CHALLENGE** In the figure at the right, if $\triangle WJZ$ is equilateral and $\angle ZWP \cong \angle WJM \cong \angle JZL$, prove that $\overline{WP} \cong \overline{ZL} \cong \overline{JM}$.

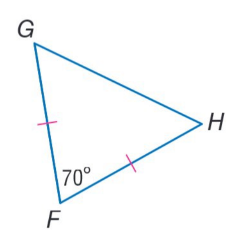


CCSS PRECISION **Determine whether the following statements are *sometimes*, *always*, or *never* true. Explain.**

46. If the measure of the vertex angle of an isosceles triangle is an integer, then the measure of each base angle is an integer.

47. If the measures of the base angles of an isosceles triangle are integers, then the measure of its vertex angle is odd.

48. **ERROR ANALYSIS** Alexis and Miguela are finding $m\angle G$ in the figure shown. Alexis says that $m\angle G = 35$, while Miguela says that $m\angle G = 60$. Is either of them correct? Explain your reasoning.



49. **OPEN ENDED** If possible, draw an isosceles triangle with base angles that are obtuse. If it is not possible, explain why not.

50. **REASONING** In isosceles $\triangle ABC$, $m\angle B = 90$. Draw the triangle. Indicate the congruent sides and label each angle with its measure.

51. **WRITING IN MATH** How can triangle classifications help you prove triangle congruence?

Standardized Test Practice

52. ALGEBRA What quantity should be added to both sides of this equation to complete the square?

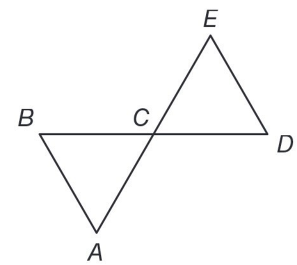
$$x^2 - 10x = 3$$

A -25 **C** 5

B -5 **D** 25

53. SHORT RESPONSE In a school of 375 students, 150 students play sports and 70 students are involved in the community service club. 30 students play sports and are involved in the community service club. How many students are *not* involved in either sports or the community service club?

54. In the figure $\overline{AE}$ and $\overline{BD}$ bisect each other at point C.



Which additional piece of information would be enough to prove that $\overline{DE} \cong \overline{DC}$?

F $\angle A \cong \angle BCA$ **H** $\angle ACB \cong \angle EDC$

G $\angle B \cong \angle D$ **J** $\angle A \cong \angle B$

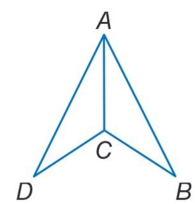
55. SAT/ACT If $x = -3$, then $4x^2 - 7x + 5 =$

A 2 **C** 20 **E** 62

B 14 **D** 42

Spiral Review

56. If $m\angle ADC = 35$, $m\angle ABC = 35$, $m\angle DAC = 26$, and $m\angle BAC = 26$, determine whether $\triangle ADC \cong \triangle ABC$. (Lesson 4-5)

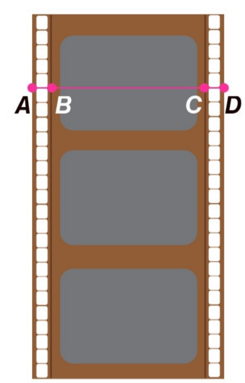


Determine whether $\triangle STU \cong \triangle XYZ$. Explain. (Lesson 4-4)

57. $S(0, 5)$, $T(0, 0)$, $U(1, 1)$, $X(4, 8)$, $Y(4, 3)$, $Z(6, 3)$

58. $S(2, 2)$, $T(4, 6)$, $U(3, 1)$, $X(-2, -2)$, $Y(-4, 6)$, $Z(-3, 1)$

59. PHOTOGRAPHY Film is fed through a traditional camera by gears that catch the perforation in the film. The distance from A to C is the same as the distance from B to D. Show that the two perforated strips are the same width. (Lesson 2-7)



State the property that justifies each statement. (Lesson 2-6)

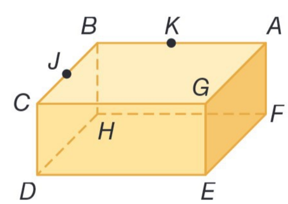
60. If $x(y + z) = a$, then $xy + xz = a$.

61. If $n - 17 = 39$, then $n = 56$.

62. If $m\angle P + m\angle Q = 110$ and $m\angle R = 110$, then $m\angle P + m\angle Q = m\angle R$.

63. If $cv = md$ and $md = 15$, then $cv = 15$.

Refer to the figure at the right. (Lesson 1-1)



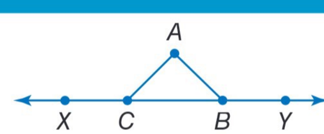
64. How many planes appear in this figure?

65. Name three points that are collinear.

66. Are points A, C, D, and J coplanar?

Skills Review

67. PROOF If $\angle ACB \cong \angle ABC$, then $\angle XCA \cong \angle YBA$.



EXPLORE 4-7 Graphing Technology Lab
Congruence Transformations

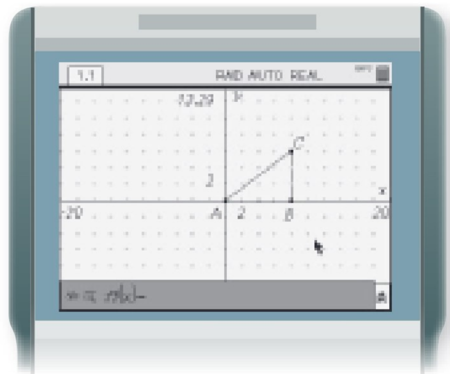
You can use TI-Nspire technology to perform *transformations* on triangles in the coordinate plane and test for congruence.

CCSS Common Core State Standards
Content Standards
G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.
G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
Mathematical Practices 5

PT

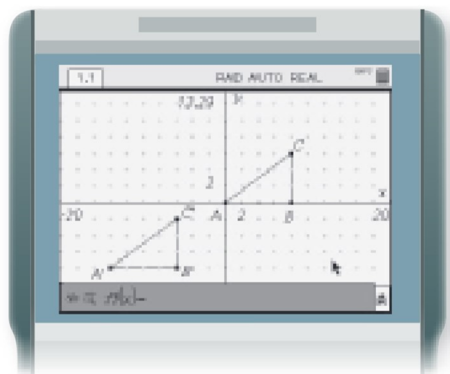
Activity 1 Translate a Triangle and Test for Congruence

Step 1 Open a new **Graphs** page. Select **Show Grid** from the **View** menu. Use the **Window/Zoom** menu to adjust the window size.

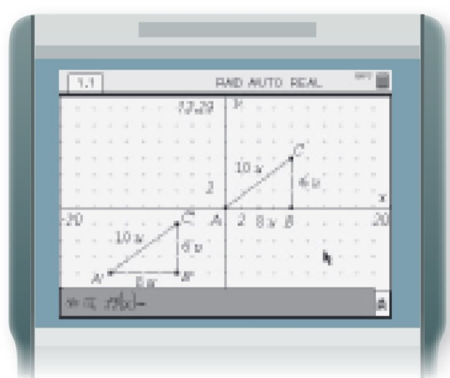


Step 2 Select **Triangle** from the **Shapes** menu and draw a right triangle with legs measuring 6 units and 8 units as shown by placing the first point at (0, 0), the second point at (8, 0), and the third point at (8, 6). Use the **Text** tool under the **Actions** menu to label the vertices of the triangle as A, B, and C.

Step 3 Select **Translation** from the **Transformation** menu. Then select $\triangle ABC$ and point A. Translate or *slide* the right triangle 8 units down and 14 units left. Label the corresponding vertices of the image as A', B', and C'.

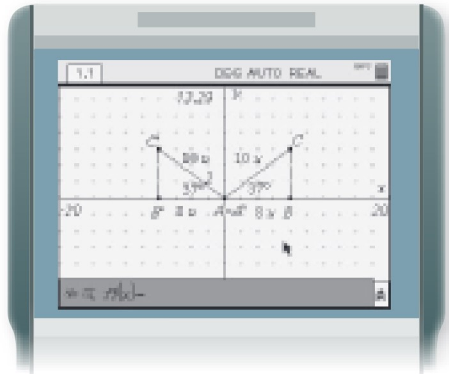


Step 4 To verify that $\triangle A'B'C'$ is congruent to $\triangle ABC$, select **Length** from the **Measurement** menu. Then select any two endpoints and press the **ENTER** key to determine the length of the segment. Repeat this for each segment of each triangle.



In addition to measuring lengths, the TI-Nspire can also be used to measure angles. This will allow you to use other tests for triangle congruence that involve angle measure.

Activity 2 Reflect a Triangle and Test for Congruence



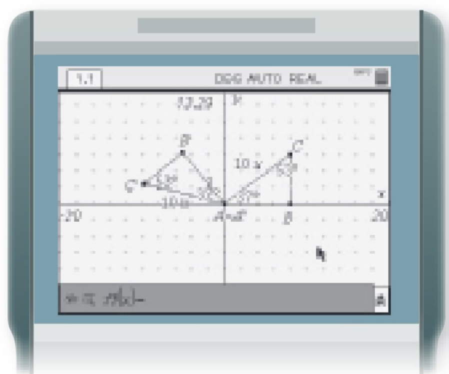
Step 1 Open a new **Graphs** page, show the grid, and redraw $\triangle ABC$ from Activity 1.

Step 2 Select **Reflection** from the **Transformation** menu. Then select $\triangle ABC$ and then the y-axis to reflect or *flip* $\triangle ABC$ in the y-axis. Label the corresponding vertices of the image as A', B', and C'.

Step 3 Use the **Angle** tool from the **Measurement** menu to find $m\angle A$ and $m\angle A'$. Use the **Length** tool from the **Measurement** menu to find AB, $A'B'$, AC, and $A'C'$.

To rotate a figure about the origin using TI-Nspire technology, use the Rotation tool to select the figure, then the point (0, 0), then draw an angle of rotation.

Activity 3 Rotate a Triangle and Test for Congruence



Step 1 Open a new **Graphs** page, show the grid, and redraw $\triangle ABC$ from Activity 1.

Step 2 Select **Rotation** from the **Transformation** menu. Then select $\triangle ABC$, select the origin, and type in a number for the angle of rotation.

Step 3 Use the **Angle** tool from the **Measurement** menu to find $m\angle A$, $m\angle A'$, $m\angle C$, and $m\angle C'$. Use the **Length** tool from the **Measurement** menu to find AC and $A'C'$.

Analyze the Results

Determine whether $\triangle ABC$ and $\triangle A'B'C'$ are congruent. Explain your reasoning.

1. Activity 1

2. Activity 2

3. Activity 3

4. Explain why $\triangle A'B'C'$ in Activity 3 does not appear to be congruent to $\triangle ABC$.

5. **MAKE A CONJECTURE** Repeat Activities 1–3 using a different triangle XYZ. Analyze your results and compare them to those found in Exercises 1–3. Make a conjecture as to the relationship between a triangle and its transformed image under a translation, reflection, or a rotation.

6. Do the measurements and observations you made in Activities 1–3 constitute a proof of the conjecture you made in Exercise 5? Explain.